

First-order quasilinear canonical representation of the characteristic formulation of the Einstein equations

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We prescribe a choice of 18 variables in all that casts the equations of the fully nonlinear characteristic formulation of general relativity in first-order quasi-linear canonical form. At the analytical level, a formulation of this type allows us to make concrete statements about existence of solutions. In addition, it offers concrete advantages for numerical applications as it now becomes possible to incorporate advanced numerical techniques for first order systems, which had thus far not been applicable to the characteristic problem of the Einstein equations, as well as in providing a framework for a unified treatment of the vacuum and matter problems. This is of relevance to the accurate simulation of gravitational waves emitted in astrophysical scenarios such as stellar core collapse.

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I. INTRODUCTION

The characteristic formulation of general relativity due to Bondi and Sachs [1, 2], in particular the approach based on a null slicing of spacetime with a transverse timelike data surface as proposed by Tamburino and Winicour [3], has been used successfully for many applications in numerical relativity. It provides a natural framework for the computation of gravitational radiation signals from isolated astrophysical sources by purely characteristic evolution [4], and as a means of extracting gravitational radiation information from $3+1$ simulations [5].

It has been used to achieve long-term stable numerical evolutions of generic single black hole spacetimes [6, 7], it has also been used to compute the behavior of matter fields around black holes [8, 9, 10], and is ideally suited to the study of gravitational radiation of black hole-neutron star mergers, the prime candidates for detection by advanced gravitational wave interferometers [11]. A modification proposed recently [12] incorporates into the formulation a partial treatment of matter fields with the specific goal of modeling the capture of a massive object such as a neutron star by a galactic-size black hole, up to the point where tidal disruptions become important. Events of this type are the predominant sources of gravitational radiation which are expected to fall in the frequency band of LISA.

Additionally, the formulation provides a unique approach to the post-merger regime of binary black hole coalescence, starting from the gravitational radiation

emitted during a white hole fission [13], as illustrated in [14, 15] in the close-limit of the binary black hole merger. It has also been used to study the nonlinear generation of waveforms in single black-hole spacetimes [16, 17], where it has shown the potential to generate a catalog of waveforms, also of interest for data analysis of space-based gravitational interferometers.

One would expect the stability properties exhibited by numerical representations of the characteristic problem [4, 5, 6, 7, 18] to reflect underlying stability properties at the analytical level, perhaps along the lines of [19, 20]. In this regard, in [21] the linearized equations are cast into a *canonical* first-order form that is suitable for the use of Duff's theorem of existence of solutions [22]. However, the choice of variables of [21] does not allow for an extension of the result to the non-linear case in any obvious manner. It is desirable to have a quasilinear formulation of the characteristic equations [5, 18] in first-order form that is to the characteristic problem what a first-order formulation is to a Cauchy problem. Such a formulation could in principle be used as the starting point to approach relevant issues of stability by means of energy estimates.

Of particular importance to the numerical implementation of the characteristic approach is that by writing the system of equations in first-order quasilinear form, conservative schemes and Godunov-type shock capturing methods [23, 24] commonly employed to solve non-linear hyperbolic systems of conservation laws (e.g. the Euler equations) can be brought to bear on the gravitational field equations [25]. High resolution shock capturing methods have been successfully used in the characteristic formulation for the evolution of matter fields [9, 10], but they have not been applied to the equations for the gravitational fields themselves, neither in the 3-dimensional case, with the equations in the form presented in Ref. [18] or in Ref. [4], nor, to the best of our knowledge, to the

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equations for the axisymmetric case [26]. The approach developed by Pons et. al [27] extends the use of special relativistic Riemann solvers, developed for special relativistic flows [28], through local coordinate transformations, thereby making them applicable to general relativistic problems as well. An obvious impediment to a unified treatment, as proposed in Pons et. al. [27], has been the lack of a first order representation of the characteristic formulation.

This leads to a subtle point deserving of clarification: true shocks do not form in the gravitational field, thus strictly speaking, shock capturing methods are necessary for the matter evolution equations only. In the presence of matter however, given that the matter fields act as sources for the metric fields, shocks and steep gradients in the matter sources lead invariably to the appearance of short-scale spatial features in the gravitational fields. The centered finite difference schemes used in the characteristic codes [4, 8, 18, 26] are not capable of resolving these short scales, and invariably generate high frequency noise. In a perverse twist, the more successful a shock capturing method is at resolving short-scale matter features, the more it compounds the problem of integrating the metric fields across these features by means of standard centered difference schemes. The effect is a classical illustration of the error one finds in advecting a pulse using a non-shock capturing centered difference method, e.g. Lax-Wendroff, where spurious oscillations appear on the trailing edge of the numerical solution. With a convergent scheme, these unwanted oscillations decrease with increasing resolution and vanish in the continuum limit, but are present in all simulations with practical grid sizes. Attempts to filter out the noise lead to a spreading of the pulse. In the problem at hand, it can lead to an unacceptable trade-off between the accuracy of the matter fields evolution vs. that of the metric fields. This effect is mentioned in recent work by Siebel et.al. [29], who perform characteristic evolution in axisymmetric stellar core collapse. In their work, the matter fields are solved via shock capturing methods [9] while the metric evolution is treated by centered, second order accurate finite difference schemes [26], with modifications along the lines of [18]. This is a difficult numerical task: in the core collapse scenario, a strong shock wave forms after bounce, where all matter fields are discontinuous. The dynamics of the collapse are correctly solved and these discontinuities are accurately tracked [29]. However, the discontinuities in the matter field lead to unavoidable discontinuities in the first derivatives of the metric fields. In the traditional scheme, second derivatives of the metric are needed to compute the gravitational radiation, and this is where numerical noise is generated, the main difficulty pointed out in [29]. A form of the equations without second order derivatives would provide a solution to this problem, and thus be particularly relevant to astrophysical problems with shocks. In the absence of such a formulation, filtering of the unwanted noise on a fixed grid has been shown not to provide suf-

ficient accuracy, and the conclusion drawn in [29] is that unless the short scale gravitational field features are adequately resolved and the resulting spurious oscillations eliminated, it is difficult to extract more accurate gravitational signals, only the main features of the signal being obtained to good accuracy.

It is generally accepted that it will be necessary to incorporate adaptive mesh refinement techniques in order to achieve well-resolved simulations with matter sources [12, 29]. Preliminary steps in these direction have been taken, with the implementation of an adaptively refined code for the model problem of Einstein-Klein-Gordon fields in spherical symmetry [30]. This task can also be more easily addressed when both the gravitational field equations and the matter equations are written in first order differential form. We should also point out another difference between the matter and gravitational field equations. Since the matter evolution equations express the conservation of the stress-energy tensor, they can be easily expressed in conservative form [9]. The gravitational field evolution equations, on the other hand, need not be put in conservative form, as they can be treated with volume preserving methods which apply equally to first order quasilinear systems and are also available [31].

Here we introduce a set of variables and auxiliary equations that casts the nonlinear gravitational field equations into a quasilinear, first-order representation of the Einstein equations in the null cone formalism that takes Duff's canonical form and provides a bridge to potential adaptations of Cauchy methods to the characteristic problem for the Einstein equations. The Tamburino-Winicour version of the Bondi-Sachs characteristic problem reduced to first-order in angular derivatives is reviewed in Section II and is taken as the starting point for the remainder of the work. In Section III the first-order quasilinear canonical characteristic form of the equations is derived. We offer concluding remarks in Section IV.

II. THE CHARACTERISTIC PROBLEM OF THE EINSTEIN EQUATIONS

As in Refs. [4, 5, 18], we use coordinates based upon a family of outgoing null hypersurfaces. We let u label these hypersurfaces, x^A ($A = 2, 3$) label the null rays and r be a surface area coordinate, such that in the $x^\alpha = (u, r, x^A)$ coordinates the metric takes the Bondi-Sachs form [1, 2]

$$ds^2 = - \left(e^{2\beta} \frac{V}{r} - r^2 h_{AB} U^A U^B \right) du^2 - 2e^{2\beta} du dr - 2r^2 h_{AB} U^B du dx^A + r^2 h_{AB} dx^A dx^B, \quad (1)$$

where h_{AB} is conformal to the metric of the sections of fixed value of r on the null slice, and $\det(h_{AB}) = \det(q_{AB})$, with q_{AB} a unit sphere metric. We define the inverse by $h^{AB} h_{BC} = \delta_C^A$. By representing ten-

sors in terms of spin-weighted variables [4], the conformal metric h_{AB} is completely encoded in the complex function $J \equiv \frac{1}{2}h_{AB}q^Aq^B$, where q^A is a complex dyad such that $q^A\bar{q}_A = -2$ and $q^Aq_A = 0$ (an overbar denotes complex conjugation). The remaining dyad component of the conformal metric, given by the real function $K = \frac{1}{2}h_{AB}q^A\bar{q}^B$, is determined by $K^2 = 1 + J\bar{J}$ as a consequence of the determinant condition. Additionally, we define $U \equiv U_Aq^A$. Angular derivatives of tensor components are in turn expressed in terms of $\bar{\partial}$ and ∂ operators [32].

The equations for the characteristic (or null cone) formulation follow from projections of the Ricci tensor normal and tangent to the null slices [2]. The resulting *main equations* arrange into a hierarchy, splitting into a set of *hypersurface equations*, which involve only derivatives on the null cone, and *evolution equations* involving derivatives with respect to the retarded time u . In particular, $R_{rr} = 0$ provides an equation for $\beta_{,r}$ in terms of J , while $R_{rA}q^A = 0$ gives $U_{,rr}$ in terms of J and β , and the trace $R_{AB}h^{AB} = 0$ yields $V_{,r}$ in terms of J , β and U . Finally, $R_{AB}q^Aq^B$ supplies the evolution equation for J . The remaining four components of the Ricci tensor vanish as a consequence of these six in the following sense. By virtue of the Bianchi identities, the component $R_{ur} = 0$ is trivially satisfied wherever the *main equations* are satisfied, whereas the remaining components $R_{uu} = 0$ and $R_{uA}q^A = 0$ are propagated radially on the null slices if they hold on a surface $r = r_0$. Thus $R_{uu} = 0$ and $R_{uA}q^A = 0$ (the *supplementary conditions*) can be viewed as constraints on the data at $r = r_0$. In the following, we ignore these constraints. We will also ignore matter source terms, although including them is straightforward (see [8]), in the interest of keeping the presentation concise.

For the present derivation, we find it convenient to start from a relatively recent representation of the characteristic formulation [18] which casts the system into first-order form in the angular derivatives, and mixed first-second-order form in the radial derivatives –as opposed to the standard, mixed-order form [4, 5]. We depart however from [18] in our choice of fundamental variables in order to facilitate the presentation in Sec. III. In this *partially reduced* form, the complete system of main equations of the characteristic formulation consists of a complex evolution equation for the conformal metric

$$2(rJ)_{,ur} - \left((1+r\tilde{W})(rJ)_{,r} \right)_{,r} = \mathcal{D} + J_H + JP_u, \quad (2)$$

namely Eq. (16) of Ref. [18], and the hierarchy of hypersurface equations and auxiliary definitions as follows:

$$\nu_{,r} = \bar{\partial}J_{,r} \quad (3)$$

$$\mu_{,r} = \bar{\partial}J_{,r} \quad (4)$$

$$\beta_{,r} = \frac{r}{8}(J_{,r}\bar{J}_{,r} - K_{,r}^2) \quad (5)$$

$$B_{,r} = \bar{\partial}\beta_{,r} \quad (6)$$

$$\begin{aligned} (r^2Q)_{,r} &= 2r^2B_{,r} - 4rB + r^2 \left[-K(k_{,r} + \nu_{,r}) \right. \\ &\quad \left. + \bar{\nu}J_{,r} + \bar{J}\mu_{,r} + \nu K_{,r} + J\bar{k}_{,r} - J_{,r}\bar{k} \right] \\ &\quad + \frac{r^2}{2K^2} [\bar{\nu}(J_{,r} - J^2\bar{J}_{,r}) + \mu(\bar{J}_{,r} - \bar{J}^2J_{,r})] \quad (7) \\ r^2U_{,r} &= e^{2\beta}(KQ - J\bar{Q}) \quad (8) \\ (r^2\tilde{W})_{,r} &= \frac{1}{2}e^{2\beta}\mathcal{R} - 1 + r\bar{\partial}U + r\partial\bar{U} + \frac{r^2}{4}(\bar{\partial}U_{,r} + \partial\bar{U}_{,r}) \\ &\quad + e^{2\beta} \left(-K(\bar{\partial}B + B\bar{B}) \right. \\ &\quad \left. + \frac{1}{2}[\bar{J}(\bar{\partial}B + B^2) + J(\bar{\partial}\bar{B} + \bar{B}^2)] \right. \\ &\quad \left. + \frac{1}{2}[(\nu - k)\bar{B} + (\bar{\nu} - \bar{k})B] \right) \\ &\quad - e^{-2\beta}\frac{r^4}{8} \left[\bar{U}_{,r}(KU_{,r} + J\bar{U}_{,r}) \right. \\ &\quad \left. + U_{,r}(K\bar{U}_{,r} + \bar{J}U_{,r}) \right] \quad (9) \end{aligned}$$

These are Eqs.(21)-(26) of [18] with the identification $\mu \equiv \bar{\partial}J$. In Eq. (2), the left-hand side is a characteristic representation of a wave operator of second differential order in r and u , and we have split the right hand-side into three terms, according to usage. The symbol P_u singles out the only term where a retarded time derivative appears in any right-hand side, namely $J_{,u}$:

$$P_u \equiv \frac{r}{K} [J_{,u}(\bar{J}_{,r}K - \bar{J}K_{,r}) + \bar{J}_{,u}(J_{,r}K - JK_{,r})] \quad (10)$$

The occurrence of this retarded time derivative in first order will be found to be critical to the developments that follow in Section III. The symbol J_H stands for the following collection of terms:

$$\begin{aligned} J_H &= \frac{e^{2\beta}}{r} \left(-K(\mu\bar{B} + 2k\bar{B} - \nu\bar{B}) + B(\bar{J}\mu + J\bar{\nu}) \right. \\ &\quad \left. + J(\bar{B}k - B\bar{k}) + J[-2K(\bar{\partial}\bar{B} + B\bar{B}) \right. \\ &\quad \left. + J(\bar{\partial}\bar{B} + \bar{B}^2) + \bar{J}(\bar{\partial}B + B^2)] \right) \\ &\quad + \frac{r^3}{2}e^{-2\beta} \left((KU_{,r} + J\bar{U}_{,r})^2 \right. \\ &\quad \left. - \frac{J}{2}[(KU_{,r} + J\bar{U}_{,r})\bar{U}_{,r} + (K\bar{U}_{,r} + \bar{J}U_{,r})U_{,r}] \right) \\ &\quad - \frac{\nu}{2}(rU_{,r} + 2U) - \frac{\mu}{2}(r\bar{U}_{,r} + 2\bar{U}) \\ &\quad + \frac{J}{2}(r\bar{\partial}U_{,r} + 2\bar{\partial}U) - \frac{J}{2}(r\partial\bar{U}_{,r} + 2\partial\bar{U}) \\ &\quad + (1-K)(r\bar{\partial}U_{,r} + 2\bar{\partial}U) - \frac{r}{2}J_{,r}(\bar{\partial}\bar{U} + \bar{\partial}U) \\ &\quad + \frac{r}{2}(\bar{U}\mu + U\nu)(J\bar{J}_{,r} - \bar{J}J_{,r}) - r\bar{U}\mu_{,r} - rU\nu_{,r} \\ &\quad + r(J_{,r}K - JK_{,r}) \left(k\bar{U} + \bar{k}U + K(\bar{\partial}\bar{U} - \bar{\partial}U) \right. \\ &\quad \left. + J\bar{\partial}\bar{U} - \bar{J}\partial U \right) - 8J(1+r\tilde{W})\beta_{,r} \quad (11) \end{aligned}$$

Any remaining terms are collected into the symbol \mathcal{D} for notational convenience:

$$\mathcal{D} = -K(\bar{\partial}rU_{,r} + 2\bar{\partial}U) + \frac{2e^{2\beta}}{r}(\bar{\partial}B + B^2) - (r\tilde{W})_{,r}J. \quad (12)$$

We also have

$$\mathcal{R} = 2K + \frac{1}{2}[\bar{\partial}(\nu - k) + \bar{\partial}(\bar{\nu} - \bar{k})] + \frac{1}{4K}(|\mu|^2 - |\nu|^2), \quad (13)$$

and use the symbol k throughout for

$$k \equiv \frac{\mu\bar{J} + J\bar{\nu}}{2K}. \quad (14)$$

The symbol \tilde{W} is simply a renaming of the original metric function V that is regular at $r = \infty$, being defined by $V \equiv r + r^2\tilde{W}$. The symbol Q is a first-order variable encoding the radial derivative of U , and is *defined* by Eq. (8), which acts as a hypersurface equation for U . In addition, ν, μ and B are first-order variables of spin weights 1, 3 and 1, respectively, used to reduce the differential order of the angular derivatives appearing in the original characteristic equations, and are defined by

$$\nu \equiv \bar{\partial}J, \quad \mu \equiv \bar{\partial}J, \quad B \equiv \bar{\partial}\beta. \quad (15)$$

With the definitions (14)-(15), Eq. (11) follows from Eq. (25) of [18]. Equations (2)-(9) as they stand contain no second-order derivatives in the retarded time or the angular coordinates. Additionally, all appearances of $U_{,r}$ and $\tilde{W}_{,r}$ in the right-hand sides represent angular derivatives and undifferentiated terms by virtue of (8) and (9). The system is still of second differential order overall, because of the presence of second-order derivatives of J , but its value resides in the fact that it exhibits remarkable numerical stability properties [18], raising the question of whether a full reduction to proper first order may further enhance the numerical stability.

III. REDUCTION OF THE TAMBURINO-WINICOUR SYSTEM TO FIRST-ORDER QUASILINEAR FORM

Our immediate goal is to write the full nonlinear equations in a quasi-linear first-order form in the strict sense, that is:

$$A^\alpha(u, r, x^A, v)v_{,\alpha} + s(u, r, x^A, v) = 0, \quad (16)$$

where v represents the set of all dependent variables, the index α runs over all spacetime coordinates and where the matrices A^α and the vector of source terms s depend on the coordinates and the undifferentiated variables v . Since all remaining second-order terms contain $J_{,r}$, and because *the only nonlinear terms in first-derivatives are quadratic and contain $J_{,r}$ as a factor* – see Eq. (10) –, this can be accomplished if an appropriate r -derivative of the fundamental field J is re-defined as an additional

fundamental variable, and there is any number of different acceptable re-definitions, one of which was used in [21]. Proceeding along the lines of Ref. [18], we define

$$H \equiv (rJ)_{,r}, \quad (17)$$

which has spin weight 2. No other radial derivatives are necessary to convert Eqs. (2)-(9) down to first-order *quasilinear* form. With this definition, the left-hand side of Eq. (2) becomes $2H_{,u} - \left((1 + r\tilde{W})H\right)_{,r}$. In the process, however, the term $J_{,u}$ in the right-hand side of Eq. (2) is promoted to the principal symbol of the system, with the consequence that the evolution equation involves the retarded time derivatives of two complex variables (H and J), instead of just one. It is unclear at this point whether Eq. (2) would determine the evolution of H or of J . (The difficulty does not arise if one linearizes the equations before reducing to first-order form, as was done in [21].) In fact, we know of no solution-generating process for the system at this point. So we proceed as follows.

In order to avoid the occurrence of the retarded-time derivative of J in the right-hand side of Eq. (2) we *define it* as an additional fundamental variable:

$$F \equiv J_{,u}, \quad (18)$$

which has spin weight 2. This is at first counter-intuitive: the equations are already first order in ∂_u , so defining the u -derivative as a new variable might not appear necessary, or even consistent. However, in the following we show that by defining this additional variable, the characteristic equations take the canonical hierarchical form needed for the existence of a solution from characteristic data [22]. With the definitions (17)-(18), the original evolution equation, Eq. (2), is interpreted as a wave equation for H , shown below as Eq. (19j). From (18) we have $(rF)_{,r} = (rJ)_{,ur}$, which yields a hypersurface equation for F , namely Eq. (19i) below. With this we can finally write the system in the form

$$J_{,r} = \frac{1}{r}(H - J), \quad (19a)$$

$$\mu_{,r} = \frac{1}{r}(\bar{\partial}H - \mu), \quad (19b)$$

$$\nu_{,r} = \frac{1}{r}(\bar{\partial}H - \nu), \quad (19c)$$

$$\beta_{,r} = \frac{r}{8}(J_{,r}\bar{J}_{,r} - K_{,r}^2) \quad (19d)$$

$$8rB_{,r} = r\mu_{,r}(\bar{H} - \bar{J}) + r\bar{\nu}_{,r}(H - J) - \frac{1}{K}[\bar{J}(H - J) + J(\bar{H} - \bar{J})]rk_{,r} \quad (19e)$$

$$\begin{aligned}
(r^2 Q)_{,r} &= 2r^2 B_{,r} - 4rB + r^2 \left[-K(k_{,r} + \nu_{,r}) \right. \\
&\quad \left. + \bar{\nu} J_{,r} + \bar{J} \mu_{,r} + \nu K_{,r} + J \bar{k}_{,r} - J_{,r} \bar{k} \right] \\
&\quad + \frac{r^2}{2K^2} [\bar{\nu}(J_{,r} - J^2 \bar{J}_{,r}) + \mu(\bar{J}_{,r} - \bar{J}^2 J_{,r})] \quad (19f)
\end{aligned}$$

$$r^2 U_{,r} = e^{2\beta} (KQ - J\bar{Q}) \quad (19g)$$

$$\begin{aligned}
(r^2 \tilde{W})_{,r} &= \frac{1}{2} e^{2\beta} \mathcal{R} - 1 + r\bar{\partial}U + r\partial\bar{U} \\
&\quad + \frac{1}{4} (\bar{\partial} [e^{2\beta} (KQ - J\bar{Q})] + \partial [e^{2\beta} (K\bar{Q} - JQ)]) \\
&\quad + e^{2\beta} \left(-K(\bar{\partial}B + B\bar{B}) \right. \\
&\quad \left. + \frac{1}{2} [\bar{J}(\bar{\partial}B + B^2) + J(\bar{\partial}\bar{B} + \bar{B}^2)] \right. \\
&\quad \left. + \frac{1}{2} [(\nu - k)\bar{B} + (\bar{\nu} - \bar{k})B] \right) \\
&\quad - \frac{e^{2\beta}}{8} [Q(K\bar{Q} - J\bar{Q}) + \bar{Q}(KQ - JQ)] \quad (19h)
\end{aligned}$$

$$2(rF)_{,r} = \left((1 + r\tilde{W})H \right)_{,r} + \mathcal{D} + J_H + JP_u, \quad (19i)$$

for the hypersurface equations and

$$2H_{,u} - [(1 + r\tilde{W})H]_{,r} = \mathcal{D} + J_H + JP_u, \quad (19j)$$

for the evolution equation, where

$$\begin{aligned}
P_u &= F(\bar{H} - \bar{J}) + \bar{F}(H - J) \\
&\quad - \left(\frac{F\bar{J} + \bar{F}J}{2K^2} \right) [(H - J)\bar{J} + J(\bar{H} - \bar{J})] \quad (20)
\end{aligned}$$

Eqs. (19b), (19c), (19e) and (19i) arise from taking an r -derivative of (15) and commuting the derivatives in the right-hand side, as usual. All the radial derivatives indicated in the right-hand side of Eqs. (19e)-(19j) can be substituted, in turn, by quantities computed previously in the hierarchy, i.e. the right-hand sides of the equations can be expressed purely in terms of the undifferentiated fundamental variables and their angular derivatives. The substitutions have been left indicated for the sake of brevity, noting only that the derivatives of k (which is not part of the hierarchy) are given by

$$\begin{aligned}
k_{,r} &= \frac{1}{2K} (\bar{J}\mu_{,r} + J\bar{\nu}_{,r} + \bar{\nu}J_{,r} + \mu\bar{J}_{,r}) - \frac{kK_{,r}}{K} \\
\bar{\partial}k &= \frac{1}{2K} (\bar{J}\bar{\partial}\mu + J\bar{\partial}\bar{\nu} + \mu^2 + \nu\bar{\nu}) - \frac{k\bar{k}}{K} \quad (21)
\end{aligned}$$

Equations (19a)-(19i) can be viewed as propagation equations along the radially outgoing null geodesics, with Eq. (19j) advancing the radial derivative of the spherical metric function J in time.

In this first-order formulation of the null cone approach, Eqs. (19a)-(19j), the boundary data at $r = r_0$ consists of the values of J , β , Q , U and \tilde{W} , with the values of μ , ν , B and F following from the boundary value

of J as per Eqs. (15) and (17). The consistency conditions, imposed at $r = r_0$, are propagated to the interior by Eqs. (19b), (19c), (19e) and (19i). The initial data for the system (19a)-(19j) at $u = u_0$ are the values of $H(r, x^A)$, representing the shear of the conformal metric of the spheres, given on the entire initial hypersurface. The conformal metric function J on the initial hypersurface follows by integration of H as per Eq. (19a), with the integration constant provided by the value of J at the boundary. Eqs. (19b) through (19i) in turn provide initial values for μ , ν , k , β , B , Q , U , \tilde{W} and F , while Eq. (19j) propagates H forward in retarded time. At this point, the process can be repeated, and the entire space-time exterior to the time-like data surface can be computed.

This solution-generating process lies at the basis of Duff's theorem of existence and uniqueness of solutions to characteristic problems—that is: problems for hyperbolic systems of equations where data is prescribed on a characteristic surface. From the analytical point of view, as it stands, the system of Eqs. (19) has the form

$$\partial_u q + N\partial_r q = L^1(\bar{\partial}q, \bar{\partial}q, \bar{\partial}w, \bar{\partial}w, q, w) \quad (22)$$

$$\partial_r w + M\partial_r q = L^2(\bar{\partial}q, \bar{\partial}q, \bar{\partial}w, \bar{\partial}w, q, w) \quad (23)$$

where $q \equiv H$, $w \equiv (J, \mu, \nu, \beta, B, U, Q, \tilde{W}, F)$, and N and M are certain matrices of dimension 2×2 and 14×2 respectively, depending on the undifferentiated variables. A trivial change of variable $F \rightarrow F - (1 + r\tilde{W})H/(2r)$ puts the system of equations (19a)-(19j) into a 18-dimensional first-order canonical quasi-linear form as defined by Duff [22], for 18 variables of which two (H) are *normal* and the remaining 16 ($J, \mu, \nu, \beta, B, U, Q, \tilde{W}, F$) are *null*, and with four complex constraints \mathcal{C}_i on the surface $r = r_0$ which are trivially preserved by the solution-generating process in the form $\partial_r \mathcal{C}_i = 0$. This means that the system (19) satisfies the conditions for Duff's theorem, and therefore existence and uniqueness follow from null and normal data in a manner analogous to Cauchy problems. This is not a trivial result, as readers should note that the same cannot be said if $J_{,u}$ is not defined as a fundamental variable.

IV. CONCLUSION

The system of equations (19a)-(19j) casts the Tamburino-Winicour version of the Bondi-Sachs characteristic initial value problem into a standard first-order quasilinear form.

A novel feature of this formulation is the introduction of a u -derivative of the 2-sphere metric, $h_{AB,u}$, as a fundamental variable. The necessity of this step arises only in the full non-linear characteristic problem, signaling the fact that the linearization and “canonization” operations (i.e. the reduction to a hierarchy as per the Bondi Sachs construction) do not commute in the case of the characteristic problem of the Einstein equations. This form of

the equations opens the possibility of further studies at the analytic level, specifically to look for the existence of estimates of the solution in terms of the data on the initial characteristic surface and the data on the surface of fixed radius r_0 .

In a much broader context, we have constructively shown here that only 18 variables are needed in order to achieve a first-order formulation of the Einstein equations suitable for numerical generation of solutions. And, most remarkably, the resulting nonlinearities are of the quasilinear type. By contrast, with regards to the Cauchy problem of the Einstein equations, Alekseenko and Arnold [33] show that as many as eight variables are actually needed in addition to the six three-metric components and the six extrinsic curvature components in order to obtain a full first-order reduction of the ADM equations [34]. This yields a total of at least 20 variables for the first-order Cauchy problem, with the drawback that the resulting nonlinearities are genuine (not of quasilinear type). In order to remove the genuine nonlinearities, all 18 space-derivatives of the three-metric must be added as fundamental variables, with the result that the generic quasilinear first-order reduction of the 3+1 Einstein equations requires a number of 30 variables. From this perspective, the fact that only 18 variables in all are actually sufficient for a first-order quasilinear solution-generating process for the Einstein equations from given data is both surprising and intriguing.

But perhaps the most important feature of the first-order formulation Eqs. (19a)–(19j) is its potential for accurately handling discontinuities in the first derivatives of the metric. This is very relevant to simulations of systems of astrophysical interest involving shock waves, such as stellar core collapse, where discontinuities in the matter fields arise and are transmitted to the first derivatives of the metric. In such systems, an accurate treatment of those discontinuities is essential to ensure the quality of the waveforms obtained. The quasilinear form of the equations is thus ideal for a unified treatment of the gravitational and matter evolution equations, with the introduction of more advanced numerical algorithms, in particular along the lines of [9, 10, 27, 29]. Work in these directions is currently in progress. Results of the application of the system of equations introduced here to the numerical characteristic effort will be reported elsewhere.

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